Rev. Mat. Apl. **25**:15-35

Revista de
Matemáticas Aplicadas

REPRESENTATION OF DIFFERENTIAL OPERATORS IN MULTIDIMENSIONAL SEPARABLE PERIODIZED COMPACTLY SUPPORTED WAVELET BASES

P.M. MORILLAS
Instituto de Matemática Aplicada San Luis
UNSL-CONICET
Ejército de los Andes 950, 5700 San Luis, Argentina
Email: morillas@unsl.ar

Abstract

The representation of linear operators in d-dimensional separable periodized compactly supported orthonormal wavelet bases is given. A closed form formula for the nonstandard matrix representation of differential operators ∂_{x_i} *and* $g(\partial_{x_1}, ..., \partial_{x_d})$*, where g is an analytic function, is obtained.*

AMS Classifications: 65D99, 47F05.

Key Words: Differential operators, matrix representation, wavelet bases.

1.- Introduction

Beylkin, Coifman and Rokhlin [2] (see also [3]) introduced the nonstandard representation of an operator and its asociated matrices in orthonormal wavelet bases. Then Beylkin [1] studied the nonstandard matrix representation of operators in one-dimensional compactly supported wavelet bases. These results were used by Beylkin and Keiser [4] in the adaptative numerical solution of evolution equation in one temporal and one spatial dimensions. The work of Beylkin was generalized by Hajji, Melkonian and Vaillancourt [8] to two dimensions using separable periodized compactly supported orthonormal wavelet bases and was used to solve partial differential equations numerically [9].

As we know, most practical problems are set in multiple dimensions. The aim of this work is to give a complete version of the matrix representation considered in [8] to possible application in the numerical solution of partial differential equations in any dimension. We state this representation in a multidimensional setting with different moments in each direction. We consider the linear differential operators ∂_{x_i} and $g(\partial_{x_1}, ..., \partial_{x_d})$, where g is an analytic function, obtaining a closed form formula for the nonstandard matrix representation. We use wavelets with vanishing moments to obtain an effectively sparse matrix representation. An **effectively sparse** matrix is one that differs from a sparse matrix by a matrix with an arbitrary small norm. In this work we consider the compactly supported wavelets constructed by Daubechies [5]. We use a periodized version of these wavelets to consider the multiresolution analysis of a function on an interval and to obtain a close form formula for the matrix representation of the differential operators.

In the rest of this Section we collect some basic facts which we need throughout the paper. Then in section 2, we review multiresolution analysis and wavelet bases of both $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^d)$ with $d \geq 2$. In section 3, we consider the standard and nonstandard representations of a linear operator introduced in [2]. In section 4, the nonstandard matrix representation of a linear operator is discussed. The nonstandard matrix representation of multidimensional differential operators ∂_{x_i} and $g(\partial_{x_1}, ..., \partial_{x_d})$, where g is an analytic function, is considered in sections 5 and 6, respectively. In section 7, some general aspects about the application to the numerical solution of partial differential equations are presented. Finally, some concluding remarks are given in section 8.

Given a function $f \in L^2(\mathbb{R}^d)$, $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$ we set

$$f_{j,k}(x) = 2^{dj/2} f(2^j x - k) \tag{1.1}$$

When we need to give the coordinates of $k = (k_1, ..., k_d)$ explicitly, we write $f_{j,k_1,...,k_d}$ instead of $f_{j,k}$. We denote with $\langle .,. \rangle$ the inner product in $L^2(\mathbb{R}^d)$. If $f_i \in \mathcal{L}^2(\mathbb{R}^d)$ for $i = 1, ..., d$, we denote with $\otimes_{i=1}^d f_i$ the tensor product of the functions f_i, i. e., $\otimes_{i=1}^d f_i(x_1, ..., x_d) = f_1(x_1)...f_d(x_d)$. Clearly, $(\otimes_{i=1}^d f_i)_{j,k} = \otimes_{i=1}^d (f_i)_{j,k}$.

In order to obtain the results considered in this paper, we introduce a product between multidimensional matrices, and suitable transformations of multidimensional matrices into ordinary matrices, and viceversa. Let P be a d-dimensional block matrix of type (m, n), i.e., a d-dimensional matrix of order m with d-dimen-sional block matrices of order n,

$$P^{k_1,...,k_d}, k_1, ..., k_d = 1, ..., m,$$

with entries

$$P^{k_1,...,k_d}_{l_1,...,l_d}, \quad l_1, ..., l_d = 1, ..., n,$$

and let D be a d-dimensional matrix of order n, with entries

$$D_{l_1,...,l_d}, \quad l_1, ..., l_d = 1, ..., n.$$

Then $P \odot D$ is a d-dimensional matrix of order m with entries given by

$$(P \odot D)_{k_1,...,k_d} = \sum_{l_1,...,l_d} P^{k_1,...,k_d}_{l_1,...,l_d} D_{l_1,...,l_d}.$$

Considering the indices $(k_1, ..., k_d)$ and $(l_1, ..., l_d)$ in a lexicographic order,we associate a vector to a d-dimensional matrix D of order n and a matrix to a d-dimensional block matrix P of type (m, n). Concretely, we associate with D the n^d-dimensional vector d with entries $D_{l_1,...,l_d}$ and with P the $m^d \times n^d$ matrix $I\!P$ which $(k_1, ..., k_d)$-row is the vector associated with $P^{k_1,...,k_d}$. Thus the vector associated with $P \odot D$ is $P \odot D$ is Pd. Conversely, given a vector of dimension n^d and a $m^d \times n^d$ matrix, we can associate to them a d-dimensional matrix of order n and a d-dimensional block matrix of type (m, n), respectively, by the inverse of the procedure just described.

A matrix C of order m is circulant, if

$$C_{i+1,j+1} = C_{i,j}, C_{i,1} = C_{i-1,m}, \text{ for i = 2, ..., m}$$

A block circulant matrix B of type (m, n) with circulant blocks, has the form

$$B = \begin{bmatrix} B_{1,1} & \cdots & B_{1,m} \\ \vdots & & \vdots \\ B_{m,1} & \cdots & B_{m,m} \end{bmatrix}.$$

where each block $B_{k,l}$, of order n, is circulant and B is block circulant.

We recall that if $A = (a_{ij})$ and $B = (b_{ij})$ are $m \times n$ and $p \times q$ matrices, respectively, then the Kronecker (or tensor, or direct) product of A and B is the $mp \times nq$ matrix $A \otimes B$ defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \vdots & \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

We have $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$. The Fourier matrix F_n of order n is the matrix with entries

$$(F_n)_{k,l} = \frac{1}{\sqrt{n}} e^{-\frac{2\pi i}{n}(k-1)(l-1)}, k, l = 1, ..., n$$

The inverse Fourier matrix F_n^* of order n is the adjoint (the conjugate transpose) of F ($F^*F = FF^* = I$).

2.- Wavelet Bases

In this section we review some aspects about wavelets. In section 2.1. we review the theory of multiresolution analysis and wavelet bases of $L^2(\mathbb{R}^d)$. In section 2.2. we discuss Daubechies's compactly supported wavelets.

2.1.- Multiresolution Analysis and Wavelet Bases of $L^2(\mathbb{R}^d)$

We begin considering the one-dimensional case.

Definition 2.1. A multiresolution analysis (1-D MRA) of $L^2(\mathbb{R})$ is a sequence of closed subspaces $(V_j)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ such that
(i) $... \subset V_{-1} \subset V_0 \subset V_1 \subset ...$
(ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.
(iii) $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$.
(iv) $f(x) \in V_j \Leftrightarrow f(x - 2^{-j}k) \in V_j$, for all $k \in \mathbb{Z}$.
(v) There exists a function $\phi \in V_0$, called a scaling function, such that the set $\{\phi_{0,k}; k \in \mathbb{Z}\}$ is an orthonormal basis of V_0.

It is important to note that a multiresolution analysis is also defined if the set $\{\phi_{0,k}; k \in \mathbb{Z}\}$ is merely a *Riesz basis* of V_0.

Since $\phi \in V_0 \subset V_1$, there exists a sequence $\{h_k\}_{k\in\mathbb{Z}}$, such that

$$\phi = \sum_k h_k \phi_{1,k} \tag{2.1}$$

Equation (2.1) is known as the *dilation equation*, the *two-scale difference equation*, or the *refinement equation*. We shall refer to it by the later name. We also have that the collection of functions $\{\phi_{j,k}; k \in \mathbb{Z}\}$ is an orthonormal basis of V_j.

Associated with V_j is the space W_j defined as the orthogonal complement of V_j in V_{j+1}, i. e., W_j is the space that satisfies

$$V_{j+1} = V_j \oplus W_j$$

From conditions (i)-(ii) of Definition 2.1 it follows that

$$L^2(\mathbb{R}) = \bigoplus_j W_j$$

A wavelet is a function ψ such that the collection of functions $\{\psi_{0,k}; k \in \mathbb{Z}\}$ is an orthonormal basis of W_0. Therefore, the collection of functions

$\{\psi_{j,k}; j,k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$. The wavelet ψ satisfies an equation similar to the refinement equation for the scaling function ϕ,

$$\psi = \sum_k g_k \phi_{1,k},$$

where

$$g_k = (-1)^k h_{-k+1}.$$

Now we discuss the d-dimensional case. We consider separable multiresolution analysis of $L^2(\mathbb{R}^d)$ (for other kind of multiresolution analysis see [10]). We begin introducing some notation.

Let $(W_j^{0,i})_{j\in\mathbb{Z}}$, be multiresolution analyses of $L^2(\mathbb{R})$ with scaling functions $\phi^{0,i}$ and wavelets $\phi^{1,i}$, where $i = 1, ..., d$. Define the spaces

$$\mathbf{V}_j = \bigotimes_{i=1}^{d} W_j^{0,i}.$$

Then, the sequence $(\mathbf{V}_j)_{j\in\mathbb{Z}}$ is a separable multiresolution analysis of $L^2(\mathbb{R}^d)$, that is, it satisfies

(1) $... \subset \mathbf{V}_{-1} \subset \mathbf{V}_0 \subset \mathbf{V}_1 ...$
(2) $\bigcup_{j\in\mathbb{Z}} \mathbf{V}_j$ is dense in $L^2(\mathbb{R}^d)$ and $\bigcap_{j\in\mathbb{Z}} \mathbf{V}_j = \{0\}$.
(3) $f(x) \in \mathbf{V}_j \Leftrightarrow f(2x) \in \mathbf{V}_{j+1}$.
(4) $f(x) \in \mathbf{V}_j \Leftrightarrow f(x - 2^{-j}k) \in \mathbf{V}_j$, for all $k \in \mathbb{Z}^d$.

The scaling function associated with $(\mathbf{V}_j)_{j\in\mathbb{Z}}$ is

$$\Phi = \bigotimes_{i=1}^{d} \phi^{0,i} \tag{2.2}$$

Since the set $\{\phi_{j,k}^{0,i}; k \in \mathbb{Z}\}$ is an orthonormal basis of $W_j^{0,i}$, then the set

$$\{\Phi_{j,k}; k \in \mathbb{Z}^d\} \tag{2.3}$$

is an orthonormal basis of $\mathbf{V}_j$. Let $\mathbf{W}_j$ denotes the orthogonal complement of $\mathbf{V}_j$ in $\mathbf{V}_{j+1}$, i. e., $\mathbf{W}_j$ is the space that satisfies

$$\mathbf{V}_{j+1} = \mathbf{V}_j \oplus \mathbf{W}_j$$

Let $E = \{0,1\}^d \setminus (0, ..., 0)$ and let $W_j^{1,i}$ be the orthogonal complement of $W_j^{0,i}$ in $W_{j+1}^{0,i}$. For $e \in E$ we set

$$\mathbf{W}_j^e = W_j^{e_1,1} \otimes ... \otimes W_j^{e_d,d},$$

and

$$\Psi^e = \phi^{e_1,1} \otimes ... \otimes \phi^{e_d,d}.$$

Then

$$\mathbf{W}_j = \bigoplus_{e\in E} \mathbf{W}_j^e, \tag{2.4}$$

$$\{\Psi^e_{j,k}; e \in E, k \in \mathbb{Z}^d\}$$

is an orthonormal basis of $hbox\mathbf{W}_j$, and

$$\{\Psi^e_{j,k}; e \in E, k \in \mathbb{Z}^d\ j \in \mathbb{Z}\}$$

is a separable orthonormal wavelet basis of $L^2(\mathbb{R}^d)$.

2.2.- Compactly Supported Wavelets with Vanishing Moments

In this section we present Daubechies's compactly supported wavelets. For the details we refer to [5].

Daubechies' scaling function satisfies the refinement equation

$$\phi = \sum_{m=0}^{L-1} h_m \phi_{1,m}, \tag{2.5}$$

and the wavelet function satisfies

$$\psi = \sum_{m=0}^{L-1} g_m \phi_{1,m}, \tag{2.6}$$

where

$$g_m = (-1)^m h_{L-m-1}, m = 0, 1, ..., L-1, \tag{2.7}$$

and

$$\int_{-\infty}^{\infty} \phi(x)dx = 1. \tag{2.8}$$

Both the scaling function ϕ and the wavelet function ψ have support in $[0, L-1]$. The coefficients $\{h_m\}_{m=0}^{L-1}$ are chosen so that $\{\psi_{j,k}; j,k \in \mathbb{Z}\}$ is an orthonormal basis and, in addition, the function ψ has M vanishing moments

$$\int_{-\infty}^{\infty} \psi(x)x^m dx = 0, m = 0, 1, ..., M-1. \tag{2.9}$$

The coefficients $\{h_m\}_{m=0}^{L-1}$ and $\{g_m\}_{m=0}^{L-1}$ in 2.5 and 2.6 are quadrature mirror filters. Once the filter $\{h_m\}_{m=0}^{L-1}$ has been chosen, it completely determines the functions ϕ and ψ. The numbers L and M are related. For the wavelets in [5], $L = 2M$. If additional conditions are imposed (see [2] for an example), then the relation can be different, but L is always even. From (1.1), (2.5) and (2.6) it follows that

$$\phi_{j,k} = \sum_{m=0}^{L-1} h_m \phi_{j+1,m+2k} \tag{2.10}$$

$$\psi_{j,k} = \sum_{m=0}^{L-1} g_m \phi_{j+1,m+2k} \tag{2.11}$$

Now we consider the d-dimensional case. For $i = 1, ..., d$, let scaling functions $\phi^{0,i}$ with refinement equations

$$\phi^{0,i} = \sum_{m=0}^{L_i-1} h_m^{0,i} \phi_{1,m}^{0,i}, \tag{2.12}$$

and wavelet functions $\phi^{1,i}$ that satisfy

$$\phi^{1,i} = \sum_{m=0}^{L_i-1} h_m^{1,i} \phi_{1,m}^{1,i} \tag{2.13}$$

where

$$h_m^{1,i} = (-1)^m h_{L_i-m-1}^{0,i}, m = 0, 1, ..., L_i - 1, \tag{2.14}$$

For $k \in \mathbb{Z}^d$ and $e \in E$, let

$$H_{m_1,...,m_d} = h_{m_1}^{0,1} ... h_{m_d}^{0,d}$$

and

$$G_{m_1,...,m_d}^e = h_{m_1}^{e_1,1} ... h_{m_d}^{e_d,d}$$

The d-dimensional separable scaling and wavelet functions are compactly supported in $\times_{i=1}^{d}[0, L_i]$ and satisfy

$$\Phi = \sum_{m_1=0}^{L_1-1} \dots \sum_{m_d=0}^{L_d-1} H_{m_1,\dots,m_d} \Phi_{1,m_1,\dots,m_d} \tag{2.15}$$

$$\Psi^e = \sum_{m_1=0}^{L_1-1} \dots \sum_{m_d=0}^{L_d-1} G^e_{m_1,\dots,m_d} \Phi_{1,m_1,\dots,m_d} \tag{2.16}$$

where $e \in E$. For each $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$ and $e \in E$ we also have

$$\Phi_{j,k_1,\dots,k_d} = \sum_{m_1=0}^{L_1-1} \dots \sum_{m_d=0}^{L_d-1} H_{m_1,\dots,m_d} \Phi_{j+1,m_1+2k_1,\dots,m_d+2k_d} \tag{2.17}$$

$$\Psi^e_{j,k_1,\dots,k_d} = \sum_{m_1=0}^{L_1-1} \dots \sum_{m_d=0}^{L_d-1} G^e_{m_1,\dots,m_d} \Phi_{j+1,m_1+2k_1,\dots,m_d+2k_d} \tag{2.18}$$

Here we use $[0,1]^d$-periodic wavelets [6]. The $[0,1]^d$-periodic scaling function and wavelets are defined by the formulae

$$\sum_{l_1 \in \mathbb{Z}} \dots \sum_{l_d \in \mathbb{Z}} \Phi_{j,k_1,\dots,k_d}(x_1 - l_1, \dots, x_1 - l_1)$$

and

$$\sum_{l_1 \in \mathbb{Z}} \dots \sum_{l_d \in \mathbb{Z}} \Psi^e_{j,k_1,\dots,k_d}(x_1 - l_1, \dots, x_1 - l_1), \quad e \in E,$$

respectively. In this case, the subspace $\mathbf{V}_j$ has dimension 2^{dj} with orthonormal basis

$$\{\Phi_{j,k_1,\dots,k_d}; k_1, \dots, k_d = 0, \dots, 2^j - 1\}, \tag{2.19}$$

whereas $\mathbf{W}_j$ has dimension $(2^d - 1)2^{dj}$ with orthonormal basis,

$$\{\Psi^e_{j,k_1,\dots,k_d}; e \in E, k_1, \dots, k_d = 0, \dots, 2^j - 1\}. \tag{2.20}$$

3.- The Standard and Nonstandard Form of an Operator

We discuss in this section the standard and nonstandard form of an operator introduced in [2] (see also [1]). Consider a multiresolution analysis

$$(\mathbf{V}_j)_{j\in\mathbb{Z}} \tag{3.1}$$

of $L^2(\mathbb{R}^d)$, constructed as in the previous section, and the orthogonal projection operators $P_j : L^2(\mathbb{R}^d) \to \mathbf{V}_j$ and $Q_j : L^2(\mathbb{R}^d) \to \mathbf{W}_j$ with $Q_j = P_{j+1} - P_j$. The standard form of an operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ in (3.1), is the representation of T as the collection of triplets,

$$T = \{A_j, \{B^j_{j'}\}_{j'\le j-1}, \{C^j_{j'}\}_{j'\le j-1}\}_{j\in\mathbb{Z}}, \tag{3.2}$$

where

$$A_j = Q_j\mathbf{T}Q_j, \tag{3.3}$$

$$B^j_{j'} = Q_j\mathbf{T}Q_{j'}, \tag{3.4}$$

$$C^j_{j'} = Q_{j'}\mathbf{T}Q_j. \tag{3.5}$$

The nonstandard form of T in (3.1), is the collection of triplets,

$$T = \{A_j, B_j, C_j\}_{j\in\mathbb{Z}}, \tag{3.6}$$

where

$$B_j = Q_j\mathbf{T}P_j, \tag{3.7}$$

$$C_j = P_j\mathbf{T}Q_j. \tag{3.8}$$

In numerical schemes, one considers a finest space $\mathbf{V}_n$ and a coarsest space $\mathbf{V}_{n-J}$, where J is the depth of the multiresolution analysis, i. e., we consider a "truncated" version of (3.1),

$$(\mathbf{V}_j)_{j=n-J}^{n} \tag{3.9}$$

The standard form of T in (3.9) is given by the set of operators

(3.10)
$$T_n = \{A_j, \{B^j_{j'}\}^{j-1}_{j'=n-J}, \{C^j_{j'}\}^{j-1}_{j'=n-J}, E^j_{n-J}, F^j_{n-J}, T_{n-J}\}_{n-J\leq j\leq n-1},$$

where

$$E^j_{n-J} = Q_j T P_{n-J},$$
$$F^j_{n-J} = P_{n-J} T Q_j,$$

(3.11)
$$T_{n-J} = P_{n-J} T P_{n-J},$$

whereas the nonstandard form of T in (3.9) is given by the set of operators

(3.12)
$$T_n = \{\{A_j, B_j, C_j\}_{n-J\leq j\leq n}, T_{n-J}\}.$$

In the sequel we only consider the nonstandard form of operators since it is a simple matter to obtain a standard form from the nonstandard form [2]. By (2.4), Q_j is the sum of the orthogonal projections Q^e_j onto $\mathbf{W}^e_j$ with $e \in E$, i. e.,

(3.13)
$$Q_j = \sum_{e\in E} Q^e_j,$$

From (3.13), the operators A_j, B_j and C_j in (3.12) are given by the sums,

(3.14)
$$A_j = \sum_{e,e'\in E} A^{e,e'}_j = \sum_{e,e'\in E} Q^e_j \mathbf{T} Q^{e'}_j,$$

(3.15)
$$B_j = \sum_{e\in E} B^e_j = \sum_{e,e'\in E} Q^e_j \mathbf{T} P_j,$$

(3.16)
$$C_j = \sum_{e\in E} C^e_j = \sum_{e\in E} P_j \mathbf{T} Q^e_j,$$

Thus, the nonstandard form (3.12) of T can be written as

(3.17)
$$T_n = \{\{A^{e,e'}_j, B^e_j, C^e_j\}_{n-J\leq j\leq n; e,e'\in E}, T_{n-J}\}.$$

4.- The Nonstandard Matrix Representation of a Linear Operator

In this section we consider the nonstandard matrix representation of a linear operator in the d-dimensional separable periodized compactly supported wavelet bases presented in section 2.2.. In what follows, the linearity of T, P_j and Q_j^e and the bases (2.19) for $\mathbf{V}_j$ and (2.20) for $\mathbf{W}_j$ are used. We also note that since the wavelets are periodic, all the matrices will be considered as 2^j-periodic.

4.1.- The Coordinate Matrices of P_j and Q_j.

If $f \in L^2(\mathbb{R}^d)$ is $[0,1]^d$-periodic, the coordinate matrix s^j of $P_j(f)$ has entries

$$s^j_{k_1,\dots,k_d} = \langle f, \Phi_{j,k_1,\dots,k_d} \rangle \tag{4.1}$$

and

$$P_j(f) = \sum_{k_1=0}^{2^j-1} \dots \sum_{k_d=0}^{2^j-1} s^j_{k_1,\dots,k_d} \Phi_{j,k_1,\dots,k_d} \tag{4.2}$$

From (3.13),

$$Q_j(f) = \sum_{e\in E} \sum_{k_1=0}^{2^j-1} \dots \sum_{k_d=0}^{2^j-1} d^{e,j}_{k_1,\dots,k_d} \Psi^e_{j,k_1,\dots,k_d}, \tag{4.3}$$

where the entries $d^{e,j}_{k_1,\dots,k_d}$ of the coordinate matrix $d^{e,j}$ of $Q_j^e(f)$ are given by

$$d^{e,j}_{k_1,\dots,k_d} = \langle f, \Psi^e_{j,k_1,\dots,k_d} \rangle \tag{4.4}$$

Since $\mathbf{V}_j = \mathbf{V}_{j-1} \oplus \mathbf{V}_{j-1}$, the d-dimensional fast wavelet transform (FWT) decomposition is obtained by lowpass and highpass filtering with downsampling:

$$s^{j-1}_{k_1,\dots,k_d} = \sum_{m_1=0}^{L_1-1} \dots \sum_{m_d=0}^{L_d-1} H_{m_1,\dots,m_d} s^j_{m_1+2k_1,\dots,m_d+2k_d} \tag{4.5}$$

$$d^{e,j-1}_{k_1,\dots,k_d} = \sum_{m_1=0}^{L_1-1} \dots \sum_{m_d=0}^{L_d-1} G^e_{m_1,\dots,m_d} s^j_{m_1+2k_1,\dots,m_d+2k_d} \tag{4.6}$$

whereas the d-dimensional inverse fast wavelet transform (IFWT) reconstruction is obtained by upsampling and filtering:

(4.7)
$$s^j_{2k_1+\lambda_1,\dots,2k_d+\lambda_d} = \sum_{m_1=0}^{M_1-1} \cdots \sum_{m_d=0}^{M_d-1} H_{2m_1+\lambda_1,\dots,2m_d+\lambda_d} s^{j-1}_{k_1-m_1,\dots,k_d-m_d}$$
$$+ \sum_{e\in E} \sum_{m_1=0}^{M_1-1} \cdots \sum_{m_d=0}^{M_d-1} G^e_{2m_1+\lambda_1,\dots,2m_d+\lambda_d} d^{e,j-1}_{k_1-m_1,\dots,k_d-m_d}$$

where $\lambda \in \{0,1\}^d$.

4.2.- The Coordinate Matrices of the Nonstandard Representation of T_n

The operators $A^{e,e'}_j$, B^e_j, C^e_j and T_j are represented by the d-dimensional matrices of type $(2^n, 2^n)$, $A^{j,e,e'}$, $B^{j,e}$, $C^{j,e}$ and T^j, respectively, with entries given by

$$A^{j,e,e',k_{d+1},\dots,k_{2d}}_{k_1,\dots,k_d} = \langle T\Psi^{e'}_{j,k_1,\dots,k_d}, \Psi^e_{j,k_{d+1},\dots,k_{2d}}\rangle \tag{4.8}$$

$$B^{j,e,k_{d+1},\dots,k_{2d}}_{k_1,\dots,k_d} = \langle T\Phi_{j,k_1,\dots,k_d}, \Psi^e_{j,k_{d+1},\dots,k_{2d}}\rangle \tag{4.9}$$

$$C^{j,e,k_{d+1},\dots,k_{2d}}_{k_1,\dots,k_d} = \langle T\Psi^e_{j,k_1,\dots,k_d}, \Phi_{j,k_{d+1},\dots,k_{2d}}\rangle \tag{4.10}$$

$$T^{j,k_{d+1},\dots,k_{2d}}_{k_1,\dots,k_d} = \langle T\Phi_{j,k_1,\dots,k_d}, \Phi_{j,k_{d+1},\dots,k_{2d}}\rangle \tag{4.11}$$

From (2.17) and (2.18), the entries of the above structures can be obtained from T^n by the formulae:

$$A^{j,e,e',k_{d+1},\dots,k_{2d}}_{k_1,\dots,k_d} =$$

$$= \sum_{m_1=0}^{L_1-1} \cdots \sum_{m_{2d}=0}^{L_{2d}-1} G^{e'}_{m_1,\dots,m_d} G^e_{m_{d+1},\dots,m_{2d}} T^{j+1,m_{d+1}+2k_{d+1},\dots,m_{2d}+2k_{2d}}_{m_1+2k_1,\dots,m_d+2k_d}, \tag{4.12}$$

$$B^{j,e,k_{d+1},\dots,k_{2d}}_{k_1,\dots,k_d} =$$

$$= \sum_{m_1=0}^{L_1-1} \dots \sum_{m_{2d}=0}^{L_{2d}-1} H_{m_1,\dots,m_d} G^e_{m_{d+1},\dots,m_{2d}} T^{j+1,m_{d+1}+2k_{d+1},\dots,m_{2d}+2k_{2d}}_{m_1+2k_1,\dots,m_d+2k_d}, \tag{4.13}$$

$$C^{j,e,k_{d+1},\dots,k_{2d}}_{k_1,\dots,k_d} =$$

$$= \sum_{m_1=0}^{L_1-1} \dots \sum_{m_{2d}=0}^{L_{2d}-1} G^e_{m_1,\dots,m_d} H_{m_{d+1},\dots,m_{2d}} T^{j+1,m_{d+1}+2k_{d+1},\dots,m_{2d}+2k_{2d}}_{m_1+2k_1,\dots,m_d+2k_d}, \tag{4.14}$$

$$T^{j,k_{d+1},\dots,k_{2d}}_{k_1,\dots,k_d} =$$

$$= \sum_{m_1=0}^{L_1-1} \dots \sum_{m_{2d}=0}^{L_{2d}-1} H_{m_1,\dots,m_d} H_{m_{d+1},\dots,m_{2d}} T^{j+1,m_{d+1}+2k_{d+1},\dots,m_{2d}+2k_{2d}}_{m_1+2k_1,\dots,m_d+2k_d}. \tag{4.15}$$

In terms of the operation $\odot$ we have

$$A^{e,e'}_j(f) = \sum_{k_1=0}^{2^j-1} \dots \sum_{k_d=0}^{2^j-1} (A^{j,e,e'} \odot d^{e',j})_{k_1,\dots,k_d} \Psi^e_{j,k_1,\dots,k_d}$$

$$B^e_j(f) = \sum_{k_1=0}^{2^j-1} \dots \sum_{k_d=0}^{2^j-1} (B^{j,e} \odot s^j)_{k_1,\dots,k_d} \Psi^e_{j,k_1,\dots,k_d}$$

$$C^e_j(f) = \sum_{k_1=0}^{2^j-1} \dots \sum_{k_d=0}^{2^j-1} (C^{j,e} \odot d^{e,j})_{k_1,\dots,k_d} \Phi_{j,k_1,\dots,k_d}$$

$$T_j(f) = \sum_{k_1=0}^{2^j-1} \dots \sum_{k_d=0}^{2^j-1} (T^j \odot s^j)_{k_1,\dots,k_d} \Phi_{j,k_1,\dots,k_d}$$

4.3.- The Calculus of T_n

In practice, the application of an operator T to a $[0,1]^d$-periodic function $f \in L^2(\mathbb{R}^d)$, is approximated by the application of the operator T_n to f. Since the associated matrix T^n may be dense, but the representation (decomposition) of T^n down the multiresolution analysis give rise to sparse matrices, to calculate $T_n(f)$ the following steps are followed:

Step 1: Obtain the coordinate matrix s^n of $P_n(f)$ whith entries $s^n_{k_1,\dots,k_d}$ given by (4.1).

Step 2: Using (4.5) and (4.6), from s^n obtain recursively the matrices s^j and $d^{e,j}$ for $j = n-1, ..., n-J$.

Step 3: Using (4.11), construct the matrix T^n.

Step 4: Using (4.12)-(4.15), construct recursively the matrices $A^{j,e,e'}$, $B^{j,e}$, $C^{j,e}$ and T^j, for $j = n-1, ..., n-J$,

Step 5: Calculate $T_n(f)$ using its nonstandard matrix,

$$(4.16)\qquad \begin{aligned} T_n(f) = \sum_{j=n-J}^{n-1} \Bigg[& \sum_{e\in E} \left(\sum_{k_1=0}^{2^j-1} \cdots \sum_{k_d=0}^{2^j-1} \tilde{d}^{e,j}_{k_1,\ldots,k_d} \Psi^e_{j,k_1,\ldots,k_d} \right) \\ & + \sum_{k_1=0}^{2^j-1} \cdots \sum_{k_d=0}^{2^j-1} \tilde{s}^j_{k_1,\ldots,k_d} \Phi_{j,k_1,\ldots,k_d} \Bigg] \end{aligned}$$

where

$$\tilde{d}^{e,j} = \sum_{e'\in E} A^{j,e,e'} \odot d^{e',j} + B^{j,e'} \odot s^j,$$

$$\tilde{s}^j = \sum_{e\in E} C^{j,e} \odot d^{e,j},$$

for $j = n-J+1, ..., n-1$, and

$$\tilde{s}^{n-J} = \sum_{e\in E} C^{n-J,e} \odot d^{e,n-J} + T^{n-J} \odot s^{n-J}.$$

The matrix coordinates s^n of the function $\tilde{f} = T_n(f)$ can be reconstructed from the coefficients $\tilde{d}^{e,j}$ and $\tilde{s}^j$ of the NS-representation of $\tilde{f}$ for $j = n-J, ..., n-1$, $e \in E$, by the following **pseudo-inverse fast wavelet transform**. First, we construct $\hat{s}^{n-J+1}$ from $\tilde{s}^{n-J}$ and $\tilde{d}^{e,n-J}$ using (4.7), and set $s^{n-J+1} = \hat{s}^{n-J+1} + \tilde{s}^{n-J+1}$. Then, for $j = n-J+1, ..., n-2$, $\hat{s}^{j+1}$ is constructed from $s^j (= \hat{s}^j + \tilde{s}^j)$ and $\tilde{d}^{e,j}$ by means of (4.7), and $s^{j+1} = \hat{s}^{j+1} + \tilde{s}^{j+1}$. Finally, for $j = n-1$, s^n is constructed from s^{n-1} and $\tilde{d}^{e,n-1}$ by means of (4.7).

Next, we use the results of this section to construct the nonstandard matrix representation of the linear differential operators $T = \partial_{x_i}$ and $T = g(\partial_{x_1}, ..., \partial_{x_d})$ where g is an analytic function. We only compute T^n because, as we saw in this section, the matrix representation of the nonstandard form of a linear operator T is completely determined by T^n.

5.- The Nonstandard Matrix Representation of ∂_{x_i}

It is convenient to introduce now the autocorrelation coefficients of $\{h_m^i\}_{m=0}^{L_i-1}$,

$$a_k^i = 2 \sum_{m=0}^{L_i-1-k} h_m^i h_{m+k}^i \quad , \; k = 1, ..., L_i - 1, \tag{5.1}$$

and the coefficients

$$r_l^i = \langle \phi_{0,l}^{0,i}, \frac{d}{dx} \phi^{0,i} \rangle \; , \; l \in \mathbb{Z}. \tag{5.2}$$

The coefficients r_l^i can be determined with the help of the following proposition (see [1]).

Proposition 5.1. (1) If the integrals (5.2) exist, then the coefficients r_l^i, $l \in \mathbb{Z}$, satisfy the following system of linear algebraic equations:

$$r_l^i = 2r_{2l}^i + \sum_{k=1}^{L_i/2} a_{2k-1}^i \left(r_{2l-2k+1}^i + r_{2l+2k-1}^i \right) \tag{5.3}$$

with

$$\begin{cases} \sum_{l=-(L_i-2)}^{L_i-2} l r_l^i = -1, \\ r_{-l}^i = -r_l^i, \\ r_l^i = 0, \text{ for } l \notin [-(L_i-2), L_i-2], \end{cases} \tag{5.4}$$

(2) If $M_i \geq 2$, where M_i is the number of vanishing moments of $\psi^{0,i}$, then equations (5.3) and (5.4) have a unique solution with $r_l^i \neq 0$ for $l = -L_i + 2, ..., L_i - 2$ and $r_{-l}^i = -r_l^i$.

Remark 5.2. Note that the range of the nonzero r_l^i is easy to see, since the scaling functions are supported in $[0, L_i - 1]$.

Remark 5.3. If $M_i = 1$, then equations (5.3) and (5.4) have a unique solution but the integrals (5.2) may not be absolutely convergent (see [1], Remark 1. for an example).

Since $\phi_{n,k_i}^{0,i}$, $i = 1, ..., d$, is an orthonormal basis, from (4.11) with $T = \partial_{x_i}$ and $j = n$, we have

(5.5)
$$(\partial_{x_i})^{n,k_{d+1},\ldots,k_{2d}}_{k_1,\ldots,k_d} = \langle \partial_{x_i}\Phi_{n,k_1,\ldots,k_d}, \Phi_{n,k_{d+1},\ldots,k_{2d}}\rangle = 2^n r^i_{k_{d+i}-k_i}\delta_{(k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_d),(k_{d+1},\ldots,k_{d+i-1},k_{d+i+1},\ldots,k_{2d})}$$

The only nonzero line of $(\partial_{x_i})^{n,k_{d+1},\ldots,k_{2d}}$ is given by

$$(\partial_{x_i})^{n,k_{d+1},\ldots,k_{2d}}_{k_{d+1},\ldots,k_{d+i-1},k_i,k_{d+i+1},\ldots,k_{2d}} = 2^n r^i_{k_{d+i}-k_i}, \quad k_i = 0,\ldots,2^n-1$$

In particular, the only nonzero line of $(\partial_{x_i})^{n,0,\ldots,0}$ is given by

$$(\partial_{x_i})^{n,0,\ldots,0}_{0,\ldots,0,k_i,0,\ldots,0} = 2^n r^i_{-k_i}, \quad k_i = 0,\ldots,2^n-1. \tag{5.6}$$

This line is the 2^n-dimensional vector

$$2^n[0, r^i_{-1}, \ldots, r^i_{-(L_i-2)}, 0, \ldots, 0, r^i_{L_i-2}, \ldots, r^i_1].$$

The matrices $(\partial_{x_i})^{n,k_{d+1},\ldots,k_{2d}}$ are obtained from $(\partial_{x_i})^{n,0,\ldots,0}$ by means of the formula:

$$(\partial_{x_i})^{n,k_{d+1},\ldots,k_{d+j-1},k_{d+j}+1,k_{d+j+1},\ldots,k_{2d}}_{k_1,\ldots,k_d} = \begin{cases} (\partial_{x_i})^{n,k_{d+1},\ldots,k_{2d}}_{k_1,\ldots,k_{j-1},k_j-1,k_{j+1},\ldots,k_d} & \text{if } k_j \neq 0 \\ (\partial_{x_i})^{n,k_{d+1},\ldots,k_{2d}}_{k_1,\ldots,k_{j-1},2^n-1,k_{j+1},\ldots,k_d} & \text{if } k_j = 0, \end{cases} \tag{5.7}$$

$j = 1,\ldots,d$.

6.- The Nonstandard Matrix Representation of $g(\partial_{x_1},\ldots,\partial_{x_d})$

Let g be an analytic function. There are two approaches to construct an approximation of the operator $g(\partial_{x_1},\ldots,\partial_{x_d})$,

(i) Compute the function g of the projections of ∂_{x_i}, $i = 1,\ldots,d$, onto $\mathbf{V_n}$,

$$g(P_n\partial_{x_1}P_n, \ldots, P_n\partial_{x_d}P_n)$$

(ii) Compute the projection of $g(\partial_{x_1},\ldots,\partial_{x_d})$ onto $\mathbf{V_n}$,

$$P_n g(\partial_{x_1},\ldots,\partial_{x_d})P_n$$

The difference between these two approaches depends on how well the absolute value square of the Fourier transforms of the scaling functions act as cutoff functions. The choice may depend on the applications [4]. Here we analyze the first approach. Because of (5.7), the $2^{dn} \times 2^{dn}$ matrix D_i associated with $(\partial_{x_i})^n$, $i = 1, ..., d$, is a block circulant matrix of type $(2^{(d-1)n}, 2^n)$ with circulant blocks. Therefore (see, e. g., [7] p. 128)

$$D_i = \left(F^*_{2^{(d-1)n}} \otimes F^*_{2^n}\right) \Lambda_i \left(F_{2^{(d-1)n}} \otimes F_{2^n}\right) \tag{6.1}$$

where Λ_i is the diagonal matrix containing the eigenvalues of D_i. Concretely,

$$\Lambda_i = \sum_{k=1}^{m} \Omega_m^{k-1} \otimes \Lambda_{i,k}$$

where $m = 2^{(d-1)n}$, Ω_m is the $m \times m$ diagonal matrix given by

$$\Omega_m = diag(1, W_m, W_m^2, ..., W_m^{m-1}), \text{ with } W_m = e^{\frac{2\pi i}{m}},$$

and $\Lambda_{i,k}$, $k = 1, ..., m$, are the diagonal matrix containing the eigenvalues of the circulant blocks of D_i. If $\Lambda^{i,k}$ is the vector containing the entries of the diagonal of $\Lambda_{i,k}$ and

$$D_{i,1} = \begin{bmatrix} \vec{s}_{i,1} \\ \vdots \\ \vec{s}_{i,2^{(d-1)n}} \end{bmatrix}$$

where $\left(\vec{s}_{i,1}, ..., \vec{s}_{i,2^{(d-1)n}}\right)$ is the first row of D_i, then

$$D_{i,1} F^*_{2^n} = \frac{1}{\sqrt{2^n}} \begin{bmatrix} \Lambda^{i,1} \\ \vdots \\ \Lambda^{i,2^{(d-1)n}} \end{bmatrix}$$

Remark 6.1. If $i \neq d$, then $D_{i,1}$ has nonzero elements only in its first column and in its $(2^{(d-i-1)n} k_i + 1)$-row, $k_i = 0, ..., 2^n - 1$. The matrix $D_{d,1}$ has nonzero elements only in its first row.

Since g is an analytic function, then $g(P_n \partial_{x_1}^{m_1} P_n, ..., P_n \partial_{x_d}^{m_d} P_n)$ has matrix representation given by the block circulant matrix with circulant blocks,

$$\begin{aligned} T^n &= g(D_1, ..., D_d) \\ &= \left(F^*_{2^{(d-1)n}} \otimes F^*_{2^n}\right) g(\Lambda_1, ..., \Lambda_d) \left(F_{2^{(d-1)n}} \otimes F_{2^n}\right) \end{aligned} \tag{6.2}$$

where $\Lambda = g(\Lambda_1, ..., \Lambda_d)$ is the $2^{dn} \times 2^{dn}$ diagonal matrix with entries given by

$$\Lambda_{p,p} = g\left((\Lambda_1)_{p,p}, ..., (\Lambda_d)_{p,p}\right),$$

$p = 1, ..., 2^{dn}$. Let

$$\left(\vec{t}_1, ..., \vec{t}_{2^{(d-1)n}}\right)$$

be the first row of T^n and set

$$T_1^n = \begin{bmatrix} \vec{t}_1 \\ \vdots \\ \vec{t}_{2^{(d-1)n}} \end{bmatrix}.$$

From (6.2),

$$T_1^n = \frac{1}{\sqrt{2^{dn}}} F_{2^{(d-1)n}} \tilde{\Lambda} F_{2^n}$$

where $\tilde{\Lambda}$ is the $2^{(d-1)n} \times 2^n$ matrix given by

$$\tilde{\Lambda} = \begin{bmatrix} \tilde{\Lambda}^1 \\ \vdots \\ \tilde{\Lambda}^{2^{(d-1)n}} \end{bmatrix}. \tag{6.3}$$

with

$$\tilde{\Lambda}^j = \left((\Lambda)_{(j-1)2^n+1,(j-1)2^n+1}, ..., (\Lambda)_{(j-1)2^n+2^n,(j-1)2^n+2^n}\right),$$

$j = 1, ..., 2^{(d-1)n}$.

Let $\tilde{\Lambda}^i$ be the matrix obtained from Λ^i by the same procedure for obtaining $\tilde{\Lambda}$ from Λ in (6.3). We have

$$\tilde{\Lambda}^i = \sqrt{2^{dn}} F^*_{2^{(d-1)n}} D_{i,1} F^*_{2^n}.$$

Then

$$\tilde{\Lambda} = g\left(\sqrt{2^{dn}} F^*_{2^{(d-1)n}} D_{1,1} F^*_{2^n}, ..., \sqrt{2^{dn}} F^*_{2^{(d-1)n}} D_{d,1} F^*_{2^n}\right).$$

Therefore,

(6.4)

$$T_1^n = \frac{1}{\sqrt{2^{dn}}} F_{2^{(d-1)n}} g\left(\sqrt{2^{dn}} F^*_{2^{(d-1)n}} D_{1,1} F^*_{2^n}, ..., \sqrt{2^{dn}} F^*_{2^{(d-1)n}} D_{d,1} F^*_{2^n}\right) F_{2^n}.$$

There are two special cases of the function $g(\partial_{x_1}, ..., \partial_{x_d})$ worth considering. Let first $g(\partial_{x_1}, ..., \partial_{x_d}) = \partial_{x_i}$. From (6.4), in this case $T_1^n = D_{i,1}$ as expected. Let now $g(\partial_{x_1}, ..., \partial_{x_d}) = g(\partial_{x_i})$. By Remark 6.1 and (6.4), if $i \neq d$ then T_1^n has nonzero elements only in its first column, and if $i = d$, then T_1^n has nonzero elements only in its first row.

7.- Application to Partial Differential Equations

In this section we collect some general observations from [4] and [9], about the application of the nonstandard form representation of an operator to the numerical solution of partial differential equations. The application of the particular representation presented in this paper with numerical experiments, will be the subject of future investigations.

To apply the nonstandard form representation of an operator to the numerical solution of a partial differential equation the following steps are considered:

Step 1: Discretize the partial differential equation to obtain an implicit difference equation with operator coefficients.
Step 2: Obtain the matrices of the operators appearing in the difference equation.
Step 3: Obtain the matrix s^n of $P_n(f_0)$, where f_0 is the initial value of the problem.
Step 4: From s^n obtain the matrices s^j and $d^{e,j}$ for $j = n-1, ..., n-J$.
Step 5: Solve the implicit difference equation by applying the operator to get $\tilde{d}^{e,j}$, $\tilde{s}^j$, for $j = n-J, ..., n-1$, $e \in E$,
Step 6: Reconstruct s^n using the pseudo-inverse fast wavelet transform.

The goal of any wavelet method to solving differential equations is essentially to use the fewest number of expansion coefficients to represent the solution since this leads to efficient numerical computations. Wavelet expansions distinguishes between smooth and shock-like behavior. In smooth regions few wavelet coefficients are needed and, in singular regions, large variations in the function require more wavelet coefficients. The wavelet expansion of functions that are solution of partial differential equations having regions of smooth, nonoscillatory behavior interrupted by a number of well-defined localized shocks or shock-like structures, have differences $d^{e,j}$ that are sparse and averages s^j that may be dense. The adaptive application of the nonstandard form representation of an operator to a function, makes use of the sparsity of the matrices $A^{j,e,e'}$, $B^{j,e}$, $C^{j,e}$ and T^j, and the differences $d^{e,j}$, $j = n-1, ..., n-J$, $e \in E$, to rapid evaluation of (4.16).

An important class of partial differential equations to which the representation presented in this paper could be applied are nonlinear evolution equations of the form:

$$u_t = \mathcal{L}u + \mathcal{N}f(u), u_0 = u(t_0, x), x \in [0,1]^d \tag{7.1}$$

with periodic boundary conditions:

$$u(t, x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_d) = u(t, x_1, ..., x_{i-1}, i1, x_{i+1}, ..., x_d)$$

Here $\mathcal{L}$ and $\mathcal{N}$ are independent of t, and f is typically nonlinear. The nonstandard form representation of an operator in wavelet bases to solve this type of equations is considered in [4] and [9] for dimensions $d = 1$ and $d = 2$, respectively.

8.- Concluding Remarks

In this paper we have presented the representation of linear operators in separable d-dimensional periodized compactly supported orthonormal wavelet bases with different moments in each direction. A closed form formula for the nonstandard matrix representation of multidimensional differential operators ∂_{x_i} and $g(\partial_{x_1}, ..., \partial_{x_d})$, where g is an analytic function, was obtained. These results can be applied to the numerical solution of partial differential equations. In particular, the use of different moments in each direction could permit us to work with differential operators more efficiently.

It is important to note that the periodization used here, is the simplest way to consider the multiresolution analysis of a function on an interval, but we might introduce an artificial singularity at the boundary. A more efficient approach would be to use wavelets on an interval [6].

References

[1] Beylkin G., On the representation of operators in bases of compactly supported wavelets, *SIAM J. Numer. Anal.* **29** (1992), 1716-1740.

[2] Beylkin G., Coifman R., Rokhlin V., Fast wavelet transforms and numerical algorithms I, *Comm. Pure Appl. Math.* **44**(2) (1991), 141–183.

[3] Beylkin G., Coifman R., Rokhlin V., Wavelets in numerical analysis, in Wavelets and their applications, M. B. Ruskai *et. al.* eds., Jones and Bartlett Books in Mathematics, Inc., Boston, (1992), 181–210.

[4] Beylkin G., Keiser J.M., On the adaptative numerical solution of nonlinear partial differential equations in wavelet bases, *J. of Computational Physics*, **132** (1997), 233–259.

[5] Daubechies I., Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl. Math.* **41** (1988), 909-996.

[6] Daubechies I., *Ten Lectures on Wavelets*, Society for Industrial and Applied Mathematics, Philadelfia, PA, (1992).

[7] Davis P.H., *Circulant Matrices*, Wiley-Interscience, New York, (1979).

[8] Hajji M.A., Melkonian S., Vaillancourt R., Representation of differential operators in wavelets basis, *J. Comp. Math. Appl.* **47** (2004), 1011–1033.

[9] Hajji M.A., Melkonian S., Vaillancourt R., Two-dimensional wavelet bases for partial differential operators and applications, *Operator Theory: Advances and Applications* **155** (2004), 219–233.

[10] Wojtaszczyk P., *A Mathematical Introduction to Wavelets*, Cambridge University Press, (1997).

[Paper received November 2005]